1st Homework sheet Model Theory

- Deadline: 22 February.
- Submit your solutions by handing them to the lecturer at the *beginning of the lecture at 14:00*.
- Good luck!

Exercise 1 We will start this first homework sheet with some definitions and a construction.

Let I be a non-empty set. A collection $\mathcal{F}$ of subsets of I is called a *filter* (on I) if:

(i) $I \in \mathcal{F}, \emptyset \notin \mathcal{F}$;

(ii) whenever $A, B \in \mathcal{F}$, then also $A \cap B \in \mathcal{F}$;

(iii) whenever $A \in \mathcal{F}$ and $A \subseteq B$, then also $B \in \mathcal{F}$.

A filter $\mathcal{U}$ is an *ultrafilter* if for any $X \subseteq I$ either $X \in \mathcal{U}$ or $X^c \in \mathcal{U}$ (here X^c is the complement of X in I).

Now suppose we have a non-empty collection $\{M_i : i \in I\}$ of L-structures and $\mathcal{F}$ is a filter on I. We can construct a new L-structure M, as follows. (To see that the following construction works one has to make a number of verifications to show that everything is well-defined: you are strongly encouraged to make these verifications on a piece of scratch paper, but you are not asked to hand them in.) Its universe is

$$\prod_{i \in I} M_i = \{f : I \to \bigcup_i M_i : (\forall i \in I)\, f(i) \in M_i\},$$

quotiented by the following equivalence relation:

$$f \sim g \quad :\Leftrightarrow \quad \{i \in I : f(i) = g(i)\} \in \mathcal{F}.$$

(Here one should check that this is indeed an equivalence relation.) If c is a constant belonging to L, then

$$c^M = [i \mapsto c^{M_i}],$$

where the expression on the right means: the equivalence class of the function $f \in \prod_{i \in I} M_i$ sending i to the interpretation of c in M_i. In addition, if g is an n-ary function symbol belonging to L and $[f_1], \ldots, [f_n] \in M$, then

$$g^M([f_1], \ldots, [f_n]) = [i \mapsto g^{M_i}(f_1(i), \ldots, f_n(i))].$$

(Here one should check that the expression on the right is independent of the choice of representatives f_i.) Finally, if R is an n-ary relation symbol belonging to L and $[f_1], \ldots, [f_n] \in M$, then we put

$$([f_1], \ldots, [f_n]) \in R^M \quad :\Leftrightarrow \quad \{\, i \in I : (f_1(i), \ldots, f_n(i)) \in R^{M_i} \,\} \in \mathcal{F}.$$

(Again, please check that the expression on the right is independent of the choice of representatives f_i.) The resulting structure is denoted by $\prod M_i/\mathcal{F}$. We will be most interested in the special case where $\mathcal{F}$ is an ultrafilter, in which case $\prod M_i/\mathcal{F}$ is called an *ultraproduct*.

(a) Show that any filter can be extended to an ultrafilter.

Hint: First use Zorn's Lemma (see the appendix to Chapter 2 of the handout) to show that any filter can be extended to one which is maximal in the inclusion ordering. Then show that such maximal filters have to be ultrafilters (for this use that if $\mathcal{F}$ is a filter on I and $A \notin \mathcal{F}$, then

$$\mathcal{G} = \{X \subseteq I : (\exists B \in \mathcal{F})\, B \cap A^c \subseteq X\}$$

defines a filter as well).

(b) Let $\{M_i : i \in I\}$ be a collection of L-structures and $\mathcal{U}$ be an ultrafilter on I.

Prove that for any formula $\varphi(x_1, \ldots, x_n)$ and for any $[f_1], \ldots, [f_n] \in \prod M_i/\mathcal{U}$ the following equivalence holds:

$$\prod M_i/\mathcal{U} \models \varphi([f_1], \ldots, [f_n]) \quad \Leftrightarrow \quad \{i \in I : M_i \models \varphi(f_1(i), \ldots, f_n(i))\} \in \mathcal{U}.$$

Hint: Argue by induction on the structure of φ and make your live easier by arguing that you only have to consider the connectives $\wedge, \neg, \exists$.

(c) Use parts (a) and (b) to give a proof of the compactness theorem for first-order logic.

Hint: Let T be a theory and assume that every finite subset of T has a model. Write I for the collection of *finite* subsets of T and pick for any $i \in I$ a model M_i such that $M_i \models \varphi$ whenever $\varphi \in i$. Prove that

$$\mathcal{F} = \{A \subseteq I : (\exists i \in I)\, (\forall j \in I)\, i \subseteq j \to j \in A\}$$

defines a filter on I and then use parts (a) and (b).